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Limit Theorems for Supercritical Branching Random Fields with Immigration*

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A measure-valued process which carries genealogical information is defined for a supercritical branching random field with immigration. This process counts the particles present at a final time whose ancestors had specified locations at given times in the past. A law of large numbers and a fluctuation limit theorem are proved for this process under a space-time scaling. The fluctuation limit is a nonstationary generalized Ornstein–Uhlenbeck process. An example of interest in transport theory and polymer chemistry is given. © 1988 Academic Press, Inc.

1. INTRODUCTION

Supercritical branching random fields with immigration are models of particle systems found in physics and biology. The model considered here is described roughly as follows. Particles appear in R^d at an initial time and immigrate into R^d at later times according to independent, homogeneous Poisson random fields in given sets. The particles live, migrate, and reproduce independently of each other according to certain laws, each particle giving birth to an average number of offspring greater than one (supercritical branching). The offspring obey similar rules, starting their migrations from the locations where their parents branched. We wish to analyze the asymptotic behavior of this model under a space-time scaling.

The process usually associated to such a model is the point measure-valued process determined by the locations of the particles present at each time; for this process we have obtained asymptotic results under various scalings for special models of the type considered here with general branching (not necessarily supercritical) [3, 13–15]. This process, however, contains no genealogical information; in particular, it does not tell us anything about the family relationship of the particles living at each time. In the

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present paper we study a process which carries certain genealogical information; this process relates to the particles present at a given time which descend from particles having specified locations in the past. More precisely, the state of this process at time t is the point measure assigning to the Borel set $A \subset R^d$ the number of particles present at a “final” time $T \geq t$ whose ancestors at time t had positions in A . We assume each particle has at least one descendant; hence all particles alive at each time are accounted for at the final time (this assumption can be made in the supercritical case only). More generally, we may define on an appropriate function space a point measure which counts the trajectories of the system up to time T lying in Borel sets of functions (this approach was introduced in [12] for the case without immigration); specializing to cylinder sets having bases at a single time and letting this time vary we obtain our process. Clearly this process has the same atom locations as the usual one referred to above for each time t , i.e., the positions of the particles present at time t , but now each atom has, instead of weight one, as in the usual case, a random weight equal to the number of descendants of the corresponding particle at the final time T .

We introduce a space-time scaling $(x, t) \rightarrow (T^b x, Tt)$, where $b > 0$ is a suitable constant (see the convergence conditions (2.6), (2.7)), and we will investigate the behavior of the scaled process for $t \in [0, 1]$ as $T \rightarrow \infty$. In order to obtain limits we must compensate the supercritical random weights of the particles by multiplying each one of them by $e^{-\alpha T(1-s)}$ (after scaling), where α is the Malthusian parameter of the underlying Bellman–Harris branching process and s is the time of birth (before scaling) of the first ancestor of the corresponding particle, and we must reduce the immigration intensity by the factor T^{-1} . We will obtain a law of large numbers and a fluctuation limit theorem, and study properties of the fluctuation limit process. These limit results resemble those for the usual process [3, 13–15], but they have different meanings, not only because the two processes are different but also because the results are due to different causes in each case. The main ingredient here, which is absent in the usual case, is the law of large numbers effect of the particle migrations given by an almost-sure invariance principle [18].

We will give results for two models, the “general” model, where the particle migration process and life-time distribution are quite general, the branching law has finite second moment, and the initial set and the spatial part of the immigration set where the particles appear are bounded; and the “special” model, where the migration process is Brownian motion, the lifetime distribution is exponential, the branching law has finite third moment, and the initial set and the spatial part of the immigration set are R^d . Corresponding to the two models we have two different methods of proof. The method for the general model is based on the almost-sure

invariance principle [18], and it shows clearly the law of large numbers effect of the particle migrations, but there are obstacles if we try to apply it without the boundedness assumption on the initial and the immigration sets (on the other hand, being able to choose these sets is useful for applications). The method for the special model involves explicit calculations that can be carried out in this case and depend on the fact that the initial set and the spatial part of the immigration set are R^d , but it hides the law of large numbers effect of the migrations.

This paper is a continuation of [7], where there was no immigration. The immigration produces several interesting new features; in particular, the generalized Langevin equation satisfied by the fluctuation limit process contains a space-time white noise term which is absent in [7] (in [7] all the randomness of the fluctuation limit comes from the initial condition). The present results, however, are not straightforward extensions of [7]; the basic ideas are similar but the immigration involves additional difficulties. A new result in [3] simplifies finding the Langevin equation for the fluctuation limit.

The basic background for this paper is found in [7]; however, we include here some preliminaries intended to make the paper partly expository; Section 2 contains these preliminaries. The models are described in Section 3 and the results in Section 4. The proofs are given in Section 5. Section 6 contains an example of interest in transport theory and polymer chemistry. Some remarks are made in Section 7 on possible extensions, on the martingale approach to tightness and on the differences between the present results and those for the usual process. A few of the results in this paper were announced in [17].

2. PRELIMINARIES

This part may be consulted as needed. Parts of Section 2b are necessary for the description of the models in Section 3.

2a. Notation

$\|\cdot\|$: Euclidean norm on R^d .

$\|\cdot\|_\infty$: supremum norm.

$\mathcal{B}(S)$: Borel σ -algebra of the topological space S .

If $\mathcal{C} \subset \mathcal{B}(R^d \times [0, \infty))$ and $t \in [0, \infty)$, then

$$\mathcal{C}_t \equiv \{x \in R^d: (x, t) \in \mathcal{C}\} \text{ and } \mathcal{C} \wedge t \equiv \{(x, s) \in \mathcal{C}: s \leq t\}.$$

λ_k : Lebesgue measure on R^k .

$\mathcal{S}(R^d)$, $\mathcal{S}'(R^d)$ ($= \bigcup_{p=0}^{\infty} \mathcal{S}'_p(R^d)$): Schwartz spaces of rapidly decreasing test functions and tempered distributions, respectively.

$\langle \cdot, \cdot \rangle$: canonical bilinear form on $\mathcal{S}'(R^d) \times \mathcal{S}(R^d)$.

$\| \cdot \|_{-p}$: dual norm on $\mathcal{S}'_p(R^d)$.

Processes X are written $\{X_t, t \in [0, T]\}$ or $\{X(t), t \in [0, T]\}$.

W^B : standard Gaussian white noise on $B \in \mathcal{B}(R^d)$, i.e., centered Gaussian random element of $\mathcal{S}'(R^d)$ with covariance functional

$$\text{Cov}(\langle W^B, \phi \rangle, \langle W^B, \psi \rangle) = \int_B \phi(x) \psi(x) dx, \quad \phi, \psi \in \mathcal{S}(R^d).$$

$D([0, T], R^d)$, $D([0, T], \mathcal{S}'(R^d))$: Skorohod spaces of right-continuous with left-limits functions.

\Rightarrow : weak convergence of probability measures.

\Rightarrow_f : weak convergence of finite-dimensional distributions.

a.s.: almost-surely.

2b. Supercritical Branching Random Motions

The following facts about branching processes can be found in, or deduced from, [1, 2].

Consider a Bellman–Harris branching process initiated by a single particle of age 0 at time 0. Suppose the particle lifetime distribution G is non-lattice and has no atom at 0. Assume the branching law $\{p_n\}_{n=0,1,\dots}$ has finite second moment, and denote m_1 and m_2 its mean and its second factorial moment, respectively. We consider here the supercritical case, i.e., $m_1 > 1$, and we assume $p_0 = 0$; hence all descendance lines are infinite. The Malthusian parameter α is the (unique) root of $m_1 \int_0^\infty e^{-\alpha t} G(dt) = 1$; in the supercritical case $\alpha > 0$.

Let Z_T designate the total number of particles present at time T . Then

$$e^{-\alpha T} Z_T \rightarrow cZ \quad \text{a.s. and in } L^2 \text{ as } T \rightarrow \infty, \quad (2.1)$$

where Z is a positive random variable such that

$$\begin{aligned} EZ &= 1, & EZ^2 &< \infty, \\ c &= (m_1 - 1)(\alpha m_1)^{-2} \int_0^\infty t e^{-\alpha t} G(dt), \end{aligned} \quad (2.2)$$

and we denote

$$\kappa = c(EZ^2)^{1/2}.$$

There exists a constant $K < \infty$ such that for all T

$$EZ_T \leq Ke^{\alpha T}, \quad EZ_T^2 \leq Ke^{2\alpha T}, \quad EZ_T^3 \leq Ke^{3\alpha T}, \quad (2.3)$$

the last inequality holding if also the third moment of $\{p_n\}$ is finite. When

G is exponential with parameter V , then

$$\begin{aligned}\alpha &= V(m_1 - 1), \quad c = 1, \quad \kappa = [m_2/(m_1 - 1)]^{1/2}, \\ EZ_T &= e^{\alpha T}, \quad EZ_T^2 = [m_2 e^{2\alpha T} - (m_2 - m_1 + 1)e^{\alpha T}]/(m_1 - 1), \\ EZ^2 &= m_2/(m_1 - 1).\end{aligned}\tag{2.4}$$

The next results on supercritical branching random motions are based on [18, 19].

Suppose the particles in the supercritical Bellman–Harris process above migrate randomly in R^d during their lifetimes in the following way. Migrations and lifetimes are independent. The initial particle starts from 0 at time 0, the offspring particles start from the locations where their parent particles branched, and the migration processes of sister particles have the same distribution, the particles on different descendance lines migrate and reproduce independently of each other and of everything in the past conditional upon the initial data of their motions. In particular, the particles may jump instantaneously at birth, and if the migrations have directions, then the offspring particles' initial directions may depend on their parent particle's last direction (e.g., the offsprings' initial directions may be distributed with radial symmetry about the parent particle's last direction). The particle trajectories are assumed to be right-continuous with left limits. Clearly the system can be translated so that it starts from any given point in R^d at any given time. The model so described is called a *supercritical branching random motion*.

The *renewal motion process* $X \equiv \{X(t), t \geq 0\}$ associated to the supercritical branching random motion above is defined the same way, starting from 0 at time 0, except that only one particle is produced at each branching. $X(t)$ is the position in R^d of the particle living at time t . We assume the increments of this process satisfy one of the following growth conditions for large h : there is a constant $K < \infty$ such that

- (a) $\sup_t E[\sup_{0 \leq s \leq h} \|X(t+s) - X(t)\| \mid \text{renewal times}] \leq Kh$,
- (b) $E[\sup_{0 \leq s \leq h} \|X(t+s) - X(t)\| \mid \text{renewal times}] \leq K[N(t+h) - N(t)]$ for each t , where $N(\cdot)$ is the renewal function,
- (c) if X has piecewise differentiable trajectories, then

$$\sup_t \sup_{0 \leq s \leq h} \|X(t+s) - X(t)\| \leq Kh.$$

Examples satisfying conditions (a), (b), and (c) are Brownian motion, random walks, and uniform linear transport processes, respectively.

We observe the supercritical branching random motion up to time $T > 0$, hence the descendance line trajectories lie in $D([0, T], R^d)$. We introduce

the space-time scaling

$$(x, t) \rightarrow (T^b x, Tt) \quad (2.5)$$

in this space, where $b > 0$ is a constant. Thus $g \in D([0, T], R^d)$ is mapped into $g_T \in D([0, 1], R^d)$ by the transformation

$$g_T(t) = T^{-b}g(Tt), \quad 0 \leq t \leq 1. \quad (2.6)$$

Our main assumption concerning the supercritical branching random motion is that the associated scaled renewal motion process X_T satisfies the convergence condition

$$X_T \Rightarrow L \equiv L(G) \quad \text{as } T \rightarrow \infty \quad (2.7)$$

in $D([0, 1], R^d)$, where L is a continuous time-homogeneous Markov process. The notation $L(G)$ emphasizes the fact that the distribution of this process in general depends on G (a limit of random walks with G -distributed waiting times illustrates this). The process X may have to be centered before it is scaled. Note that $X_T(0) = L(0) = 0$. Other technical conditions and details about this model are found in [18].

The supercritical branching random motion is defined on a probability space (Ω, \mathcal{F}, P) . Its *empirical distribution* at time T is the random Borel probability measure P_T on $D([0, 1], R^d)$ given by

$$P_T(\omega, \mathcal{A}) = N^T(\omega, \mathcal{A})/Z_T(\omega), \quad \omega \in \Omega, \mathcal{A} \in \mathcal{B}(D([0, 1], R^d)),$$

where $Z_T(\omega)$ was defined above and $N^T(\omega, \mathcal{A})$ is the number of particles present at time T whose ancestry line trajectories scaled by (2.6) lie in \mathcal{A} . The empirical distribution obeys the following almost-sure invariance principle [18]:

THEOREM 2.1. *Under the assumptions above,*

$$P_T(\omega, \cdot) \Rightarrow \tilde{L} \quad \text{as } T \rightarrow \infty$$

for P -almost all ω , where $\tilde{L} = L(\tilde{G})$, with $\tilde{G}(dt) = m_1 e^{-\alpha t} G(dt)$.

Thus, for almost every realization (ω) of the supercritical branching random motion the empirical distribution converges weakly to the distribution of a (non-random) process \tilde{L} , which is like the limit process of the renewal motion process, i.e., L in (2.7), with the waiting time distribution G replaced by \tilde{G} . We remark that this result is basically a strong law of large numbers. Applications of this theorem are found in [15, 18].

When $\mathcal{A} \in \mathcal{B}(D([0, 1], R^d))$ is a cylinder with base $A \in \mathcal{B}(R^d)$ at time $t \in [0, 1]$ and the initial position of the first particle is $x \in R^d$, we denote

$$N^T(\mathcal{A}) = \hat{N}_{x,t}^T(A);$$

therefore $\hat{N}_{x,t}^T(A)$ is the number of particles present at time T whose ancestors at time Tt had positions in $T^b A + x$ (the scaling (2.6) is applied to the increments of the descendance line trajectories from their initial position x). $\hat{N}_{x,t}^T(\cdot)$ is a finite random point measure on R^d ; hence we may write $\langle \hat{N}_{x,t}^T, \phi \rangle = \int_{R^d} \phi(y) \hat{N}_{x,t}^T(dy)$, $\phi \in \mathcal{S}(R^d)$.

In the forthcoming we denote by $\{\mathcal{T}_t\}$ the semigroup of the Markov process \tilde{L} in Theorem 2.1, and Z^x the limit in (2.1) for an initial particle starting from x . From (2.1) and Theorem 2.1 we have

COROLLARY 2.2. *For each $\phi \in \mathcal{S}(R^d)$,*

(a) $e^{-\alpha T} \langle \hat{N}_{x,t}^T, \phi \rangle \rightarrow c Z^x \mathcal{T}_t \phi(x)$ *a.s. and in L^2 as $T \rightarrow \infty$.*

(b) *If the particle migration process is standard Brownian motion starting from the position where the parent branched, then*

$$e^{-\alpha T} \langle \hat{N}_{x,t}^T, \phi \rangle \rightarrow c Z^x \int_{R^d} \phi(y) \frac{e^{-\|y-x\|^2/2t}}{(2\pi t)^{d/2}} dy \quad a.s.$$

and in L^2 as $T \rightarrow \infty$. ($c = 1$ if the lifetime distribution is exponential).

In case (b), (2.7) holds trivially with $b = \frac{1}{2}$ in (2.6), and $L = \tilde{L} =$ standard Brownian motion.

2c. Generalized Langevin Equations

Let $\mathcal{W} \equiv \{\mathcal{W}(t), t \geq 0\}$ be an $\mathcal{S}'(R^d)$ -Wiener process, i.e., a continuous, centered, Gaussian $\mathcal{S}'(R^d)$ -valued process whose covariance functional has the form

$$\text{Cov}(\langle \mathcal{W}(s), \phi \rangle, \langle \mathcal{W}(t), \psi \rangle) = \int_0^{s \wedge t} \langle Q_u \phi, \psi \rangle du, \\ s, t \geq 0, \phi, \psi \in \mathcal{S}(R^d),$$

where the operators $Q_u: \mathcal{S}(R^d) \rightarrow \mathcal{S}'(R^d)$ are linear, continuous, symmetric, and positive for each u , and the function $u \rightarrow \langle Q_u \phi, \psi \rangle$ is right-continuous with left limits for each $\phi, \psi \in \mathcal{S}(R^d)$. Hence $\mathcal{W}(0) = 0$ and \mathcal{W} has independent, but not necessarily stationary, increments. Let $\mathcal{H}: \mathcal{S}(R^d) \rightarrow \mathcal{S}(R^d)$ be a continuous linear operator and \mathcal{H}^* its adjoint. A continuous, centered $\mathcal{S}'(R^d)$ -valued process $X \equiv \{X(t), t \geq 0\}$ is said to satisfy the *generalized Langevin equation*

$$dX = \mathcal{H}^* X dt + d\mathcal{W} \quad (2.8)$$

if

$$\langle X(t), \phi \rangle = \langle X(0), \phi \rangle + \int_0^t \langle X(s), \mathcal{H} \phi \rangle ds + \langle \mathcal{W}(t), \phi \rangle, \quad t \geq 0,$$

holds for each $\phi \in \mathcal{S}(R^d)$. Solutions of generalized Langevin equations are called *generalized Ornstein–Uhlenbeck processes*. According to our definition these processes are in general not stationary. The stationary case has been treated extensively by Itô [21]. The following result is proved in [3].

THEOREM 2.3. *Let X be a continuous, centered, Gaussian $\mathcal{S}'(R^d)$ -valued process whose covariance functional*

$$K_X(s, \phi; t, \psi) \equiv \text{Cov}(\langle X(s), \phi \rangle, \langle X(t), \psi \rangle), \quad \phi, \psi \in \mathcal{S}(R^d)$$

satisfies the condition

$$K_X(s, \phi; t, \psi) = K_X(s, \phi; s, \mathcal{R}_{t-s}\psi), \quad s \leq t, \quad (2.9)$$

where $\{\mathcal{R}_t\}$ is a strongly continuous semigroup of continuous linear operators on $\mathcal{S}(R^d)$ whose infinitesimal generator \mathcal{H} is continuous from $\mathcal{S}(R^d)$ into itself. Then X is a Markov process and it obeys (2.8), where \mathcal{W} is a continuous $\mathcal{S}'(R^d)$ -valued Gaussian process with covariance functional

$$\begin{aligned} & \text{Cov}(\langle \mathcal{W}(s), \phi \rangle, \langle \mathcal{W}(t), \psi \rangle) \\ &= K_X(s, \phi; s, \psi) - K_X(0, \phi; 0, \psi) \\ & \quad - \int_0^s [K_X(u, \mathcal{H}\phi; u, \psi) + K_X(u, \phi; u, \mathcal{H}\psi)] du, \quad s \leq t. \end{aligned}$$

In addition, for each $\phi \in \mathcal{S}(R^d)$ the process

$$\langle X(t), \phi \rangle - \langle X(0), \phi \rangle - \int_0^t \langle X(s), \mathcal{H}\phi \rangle ds, \quad t \geq 0$$

is a square-integrable martingale (with respect to the filtration generated by the process $\{X(t)\}$) with increasing process $E\langle \mathcal{W}(t), \phi \rangle^2$, $t \geq 0$.

Remark. The martingale assertion is an easy consequence of the independence of the increments of \mathcal{W} . If $u \rightarrow K_X(u, \phi; u, \phi)$ is continuously differentiable, then \mathcal{W} is an $\mathcal{S}'(R^d)$ -Wiener process with

$$\langle \mathcal{Q}_u \phi, \psi \rangle = \frac{d}{du} K_X(u, \phi; u, \psi) - K_X(u, \mathcal{H}\phi; u, \psi) - K_X(u, \phi; u, \mathcal{H}\psi).$$

We recall that $\{\mathcal{R}_t\}$ and \mathcal{H} satisfy

$$\mathcal{R}_t \phi - \phi = \int_0^t \mathcal{R}_s \mathcal{H} \phi ds, \quad \phi \in \mathcal{S}(R^d). \quad (2.10)$$

3. MODELS

We describe here in detail the supercritical branching random field with immigration in R^d . At time 0 the particles are distributed according to a homogeneous Poisson random field with intensity $\gamma \geq 0$ in a set $B \in \mathcal{B}(R^d)$. Particles from an external source immigrate into R^d according to a

homogeneous space-time Poisson random field with intensity $\beta \geq 0$ in a set $\mathcal{C} \in \mathcal{B}(R^d \times [0, \infty))$. The two Poisson fields are independent. Each initial and each immigrant particle generates an independent supercritical branching random motion satisfying the conditions in Section 2b; in particular, the associated renewal motion process obeys the convergence condition (2.7).

We observe the system up to time $T > 0$, the final time, and applying the scaling (2.5) we denote $\hat{N}_t^T(A)$ the number of particles present at time T such that their ancestors at time Tt , $t \in [0, 1]$, had positions in $T^b A$, $A \in \mathcal{B}(R^d)$. (Notice that under the scaling the initial Poisson intensity and the spatial part of the immigration Poisson intensity are multiplied by the factor T^{bd}). Hence \hat{N}_t^T is a random point measure on R^d whose atoms are the (scaled) locations of the particles present at (scaled) time Tt , each atom having a random weight equal to the number of descendants of the corresponding particle at time T . We replace the immigration intensity β by β/T and we further weight each atom of \hat{N}_t^T by $e^{-\alpha T(1-s)}$, where α is the Malthusian parameter and s is the (unscaled) time of birth of the first ancestor of the corresponding particle. We designate N_t^T the random point measure so obtained. Our objective is to study the behavior of the process $N^T \equiv \{N_t^T, t \in [0, 1]\}$, as $T \rightarrow \infty$.

We will consider two models, the "general" model, which is as described above, with the initial and the immigration sets B and $\mathcal{C} \wedge t$ (for each t) being bounded (see, however, the remark after Theorem 4.2), and the "special" model, where the particle migration process is standard Brownian motion starting from the position where the parent branched, the lifetime distribution is exponential with parameter V and the branching law has finite third moment; in the special case B and $\mathcal{C} \wedge t$ may be unbounded, and we will consider the case $B = R^d$ and $\mathcal{C} = R^d \times C$, $C \in \mathcal{B}([0, \infty))$. For the special model the convergence condition (2.7) is trivially satisfied with $b = \frac{1}{2}$ in the scaling (2.6) and $L = \tilde{L} =$ standard Brownian motion. We will study the behavior of the process N^T and its fluctuation $N^T - EN^T$ as $T \rightarrow \infty$ for both models. For the special model N^T can be realized in $D([0, 1], \mathcal{S}'(R^d))$. For the general model N^T takes values in $\mathcal{S}'(R^d)$ because it is a random finite measure for each t , due to the boundedness of the initial and immigration sets and the conditions on G ; it seems intuitively N^T should also be realizable in $D([0, 1], \mathcal{S}'(R^d))$.

4. RESULTS

We have denoted by $\{\mathcal{T}_t\}$ the semigroup of the Markov process \tilde{L} in Theorem 2.1. Let us designate \mathcal{G} and $p_t(x, dy)$ its infinitesimal generator and its transition probability, respectively. In addition we assume $\{\mathcal{T}_t\}$ and

\mathcal{G} satisfy the conditions of Theorem 2.3. For notational consistency we put $\mathcal{T}_t = 0$ and $p_t = 0$, for $t < 0$.

The asymptotic behavior of the general model is given by the following two theorems.

THEOREM 4.1 (Law of large numbers). *For each $t \in [0, 1]$ and $\phi \in \mathcal{S}'(R^d)$,*

$$T^{-bd} \langle N_t^T, \phi \rangle \rightarrow c \left[\gamma \int_B \mathcal{T}_t \phi(x) dx + \beta \int \int_{\mathcal{G}} \mathcal{T}_{t-r} \phi(x) dx dr \right]$$

in L^2 as $T \rightarrow \infty$, where c is given in (2.2).

Let us denote by

$$M_t^T = T^{-bd/2} (N_t^T - EN_t^T), \quad 0 \leq t \leq 1,$$

the normalized fluctuation, and define the process $M^T \equiv \{M_t^T, t \in [0, 1]\}$.

THEOREM 4.2 (Fluctuation limit). *$M^T \Rightarrow_f M$ as $T \rightarrow \infty$, where $M \equiv \{M_t, t \in [0, 1]\}$ is a centered Gaussian $\mathcal{S}'(R^d)$ -valued process with covariance functional*

$$\begin{aligned} & \text{Cov}(\langle M_s, \phi \rangle, \langle M_t, \psi \rangle) \\ &= \kappa^2 \left[\gamma \int_B \mathcal{T}_s \phi(x) \mathcal{T}_t \psi(x) dx + \beta \int \int_{\mathcal{G}} \mathcal{T}_{s-r} \phi(x) \mathcal{T}_{t-r} \psi(x) dx dr \right], \\ & \quad \phi, \psi \in \mathcal{S}'(R^d), \quad (4.1) \end{aligned}$$

where κ is given in (2.2)

Remark. The boundedness of B and $\mathcal{G} \wedge t$ is used in the proofs of Theorems 4.1 and 4.2 for taking the limit $T \rightarrow \infty$ inside certain integrals, and also for studying properties of the process M (Theorem 4.3). To this end it would suffice that these sets had finite Lebesgue measures. However, the boundedness of these sets also ensures that the process N^T takes values in $\mathcal{S}'(R^d)$, since $N_t^T(A) < \infty$ for bounded A ; this is needed for the fluctuation limit. Without the boundedness assumption it could happen that $N_t(A) = \infty$ for some t and bounded A due to the generality of this model (perhaps the growth conditions on the renewal motion process preclude this).

We have observed the system up to time T and we introduced the scaling (2.5), transforming the time interval into $[0, 1]$ for the scaled process. But we may also observe the system up to time aT , $a > 1$, and use the same scaling, transforming the time interval into $[0, a]$ for the scaled process. Hence we may extend our previous results to $[0, a]$ for any $0 < a < \infty$, by counting the particles at the final time aT , and changing the exponential weights accordingly, provided the convergence condition (2.7) holds for the extended intervals. In this case the limit fluctuation process M is defined

for all $t \geq 0$. Therefore, we shall study the properties of the centered Gaussian $\mathcal{S}'(R^d)$ -valued process $M \equiv \{M_t, t \geq 0\}$ with covariance functional given by (4.1).

THEOREM 4.3 (Properties of the fluctuation limit process M). (1) M has the stochastic integral representation

$$\langle M_t, \phi \rangle = \kappa \left[\gamma^{1/2} \int_B \mathcal{T}_t \phi(x) W_1(dx) + \beta^{1/2} \int \int_{\mathcal{C}} \mathcal{T}_{t-r} \phi(x) W_2(dx, dr) \right],$$

$$\phi \in \mathcal{S}'(R^d),$$

where W_1 and W_2 are independent standard Gaussian white noise measures on R^d and R^{d+1} , respectively (see, e.g., [33] for the definitions and properties of these Itô-type multiparameter stochastic integrals).

(2) The covariance functional of M can be written

$$\text{Cov}(\langle M_s, \phi \rangle, \langle M_t, \psi \rangle) = \int_{R^d} \int_{R^d} \phi(y) \psi(z) K(s, dy; t, dz),$$

where

$$K(s, dy; t, dz) = \kappa^2 \left[\gamma \int_B p_s(x, dy) p_t(x, dz) dx \right. \\ \left. + \beta \int \int_{\mathcal{C}} p_{s-r}(x, dy) p_{t-r}(x, dz) dx dr \right]$$

is the covariance measure. For $t > 0$, M_t is induced by an ordinary Gaussian random field $\tilde{M}_t(y)$ if $p_s(x, dy)$ has a density $p_s(x, y)$ for $s = t$ and for $s = t - r$ for Lebesgue-almost all $r < t$ such that \mathcal{C}_r has positive Lebesgue measure, and these densities satisfy

$$\int_B p_t(x, y)^2 dx < \infty \quad \text{and} \quad \int \int_{\mathcal{C}} p_{t-r}(x, y)^2 dx dr < \infty.$$

In this case $\tilde{M}_t(y)$ is given by

$$\tilde{M}_t(y) = \kappa \left[\gamma^{1/2} \int_B p_t(x, y) W_1(dx) + \beta^{1/2} \int \int_{\mathcal{C}} p_{t-r}(x, y) W_2(dx, dr) \right],$$

where W_1 and W_2 are as in (1). \tilde{M}_t is norm-continuous if and only if the covariance kernel

$$k(t, y; t, z) = \kappa^2 \left[\gamma \int_B p_t(x, y) p_t(x, z) dx \right. \\ \left. + \beta \int \int_{\mathcal{C}} p_{t-r}(x, y) p_{t-r}(x, z) dx dr \right]$$

is jointly continuous in y and z . (See [28] for the connection between generalized and ordinary random fields, and the norm continuity of \tilde{M}_t).

(3) M is Markovian and satisfies the generalized Langevin equation

$$\begin{aligned} dM &= \mathcal{G}^* M dt + \kappa \beta^{1/2} d\mathcal{W}^{\mathcal{C}} \\ M_0 &= \kappa \gamma^{1/2} W^B, \end{aligned}$$

where W^B is standard Gaussian white noise on B and $\mathcal{W}^{\mathcal{C}} \equiv \{\mathcal{W}_t^{\mathcal{C}}, t \geq 0\}$ is an $\mathcal{S}'(R^d)$ -Wiener process with covariance functional

$$\text{Cov}(\langle \mathcal{W}_s^{\mathcal{C}}, \phi \rangle, \langle \mathcal{W}_t^{\mathcal{C}}, \psi \rangle) = \int_0^{s \wedge t} \int_{\mathcal{C}_r} \phi(x) \psi(x) dx dr,$$

(i.e., $d\mathcal{W}^{\mathcal{C}}$ is a standard space-time Gaussian white noise on \mathcal{C}). Moreover, $\mathcal{W}^{\mathcal{C}}$ is an $\mathcal{S}'_p(R^d)$ -valued square-integrable martingale with respect to the filtration generated by M , with $p > d/4$ (this depends on the norm on $\mathcal{S}'_p(R^d)$) and W^B and $\mathcal{W}^{\mathcal{C}}$ are independent. (We assume \mathcal{C} is sufficiently regular so that $r \rightarrow \int_{\mathcal{C}_r} \phi(x) \psi(x) dx$ is right-continuous with left limits.)

(4) M can also be expressed as the “evolution” solution of the Langevin equation, i.e.,

$$\begin{aligned} \langle M_t, \phi \rangle &= \kappa \gamma^{1/2} \langle W^B, \mathcal{T}_t \phi \rangle + \kappa \beta^{1/2} \int_0^t \langle d\mathcal{W}_r^{\mathcal{C}}, \mathcal{T}_{t-r} \phi \rangle, \\ t &\geq 0, \phi \in \mathcal{S}(R^d) \end{aligned}$$

(since $\mathcal{W}^{\mathcal{C}}$ is an $\mathcal{S}'_p(R^d)$ -valued square-integrable martingale, the convolution-type stochastic integral is well defined [6, 25]).

(5) M has a strongly continuous version, i.e., there is an integer $p > 0$ such that $\{M_t, t \geq 0\}$ is $\|\cdot\|_{-p}$ -continuous a.s.

(6) In general M_t is not homogeneous (i.e., distribution invariant under spatial translations). However, if $B = R^d$ and $\mathcal{C} = R^d \times C$, where $C \in \mathcal{B}([0, \infty))$, and $p_*(x, dy) = p_*(0, dy - x)$, then, assuming M_t is well defined, M_t is homogeneous with spectral measure

$$\sigma_t(d\lambda) = \kappa^2 \left[\gamma |\hat{p}_t(\lambda)|^2 + \beta \int_{\mathcal{C}} |\hat{p}_{t-r}(\lambda)|^2 dr \right] d\lambda, \quad t \geq 0,$$

where $\hat{p}_t(\lambda)$ is the characteristic function of $p_t(0, dy)$. (See [10] for spectral measures.)

(7) If the covariance measure $K(t, dy; t, dz)$ given in (2) converges weakly as $t \rightarrow \infty$ to a non-zero measure $K(dy, dz)$, then $M_t \Rightarrow M_\infty$ as $t \rightarrow \infty$, where M_∞ is a centered, Gaussian $\mathcal{S}'(R^d)$ -valued random variable with covariance measure $K(dy, dz)$.

If $\{\mathcal{T}_t\}$ is a Brownian semigroup, Theorem 4.3 holds also for $B = R^d$ and $\mathcal{C} = R^d \times C$, $C \in \mathcal{B}([0, \infty))$. In the next theorem we consider properties of the process M in this case. We include some of the assertions of

Theorem 4.3 due to the special form they take in this case, and the $\|\cdot\|_{-p}$ -continuity because the proof is somewhat different.

THEOREM 4.4. *Suppose the process \tilde{L} in Theorem 2.1 is Brownian motion with variance parameter σ^2 , and the initial and immigration sets are $B = R^d$ and $\mathcal{C} = R^d \times C$, $C \in \mathcal{B}([0, \infty))$, respectively. Then the process $M \equiv \{M_t, t \geq 0\}$ has the following properties:*

(1) *M has covariance kernel*

$$k(s, y; t, z) = \kappa^2 \left[\gamma e^{-\|y-z\|^2/2\sigma^2(t+s)} (2\pi\sigma^2(t+s))^{-d/2} \right. \\ \left. + \beta \int_{C \cap [0, s]} e^{-\|y-z\|^2/2\sigma^2(t+s-2r)} \right. \\ \left. \times (2\pi\sigma^2(t+s-2r))^{-d/2} dr \right], \quad s \leq t,$$

and for each $0 < t < \infty$, M_t is induced by an ordinary norm-continuous Gaussian random field.

(2) *M is Markovian and obeys the generalized Langevin equation*

$$dM = (\sigma^2/2) \Delta M dt + \kappa\beta^{1/2} d\mathcal{W}^{\mathcal{C}},$$

with M_0 and $\mathcal{W}^{\mathcal{C}}$ as in Theorem 4.3 (3).

(3) *There is an integer $p > 0$ such that M_t is $\|\cdot\|_{-p}$ -continuous a.s. for all $t \geq 0$.*

(4) *M_t is homogeneous with spectral measure*

$$\sigma_t(d\lambda) = (\kappa^2/\sigma^2) \left[\gamma e^{-t\|\lambda\|^2} + \beta \int_{C \cap [0, t]} e^{-(t-r)\|\lambda\|^2} dr \right] d\lambda, \quad t \geq 0.$$

(5) *When $C = [0, \infty)$ and $d \geq 3$, $M_t \Rightarrow M_\infty$ as $t \rightarrow \infty$, where M_∞ is a centered, Gaussian $\mathcal{S}'(R^d)$ -valued random variable with covariance functional*

$$\text{Cov}(\langle M_\infty, \phi \rangle, \langle M_\infty, \psi \rangle) \\ = (\kappa^2\beta\Gamma(d/2 - 1)/4\pi\sigma^2) \int_{R^d} \int_{R^d} \phi(x) \psi(y) \|x - y\|^{-d+2} dx dy;$$

where Γ is the gamma function. M_∞ is homogeneous with spectral measure

$$\sigma_\infty(d\lambda) = (\kappa^2\beta/\sigma^2 \|\lambda\|^2) d\lambda;$$

hence M_∞ is not induced by an ordinary random field, since σ_∞ is not a finite measure (see, e.g., [27]). M_∞ depends only on the immigration.

(6) When $C = [0, \infty)$, M can be represented as $M = \kappa(\gamma^{1/2}M^I + \beta^{1/2}M^{II})$, where M^I and M^{II} are independent processes which are self-similar under the transformations $\langle a^{-d/2}M_{at}^I, \phi(\cdot/a^{1/2}) \rangle$ and $\langle a^{-d/2-1}M_{at}^{II}, \phi(\cdot/a^{1/2}) \rangle$, respectively, for any $a > 0$.

Remark. Theorem 4.4 does not claim convergence of M^T to M with $B = R^d$ and $\mathcal{C} = R^d \times C$.

The next theorems give the asymptotic behavior of the special model (with $B = R^d$ and $\mathcal{C} = R^d \times C$, $C \in \mathcal{B}([0, \infty))$). In this case α , c , and κ are given in (2.4).

THEOREM 4.5 (Law of large numbers). *For each $t \in [0, 1]$ and $\phi \in \mathcal{S}'(R^d)$,*

$$T^{-d/2} \langle N_t^T, \phi \rangle \rightarrow [\gamma + \beta \lambda_1(C \cap [0, t])] \int_{R^d} \phi(x) dx$$

in L^2 as $T \rightarrow \infty$.

In this case the fluctuation process M^T takes the form

$$M_t^T = T^{-d/4} [N_t^T - T^{d/2}(\gamma + \beta \lambda_1(C \cap [0, t])) \lambda_d], \quad 0 \leq t \leq 1.$$

THEOREM 4.6 (Fluctuation limit). *The process M^T can be realized in $D([0, 1], \mathcal{S}'(R^d))$, and $M^T \Rightarrow_f M$ as $T \rightarrow \infty$, where $M \equiv \{M_t, t \geq 0\}$ is a centered, Gaussian $\mathcal{S}'(R^d)$ -valued process which has the properties given in Theorem 4.4, with $\sigma^2 = 1$.*

Remark. If in addition $\{M^T\}_{T \geq 1}$ is relatively weakly compact, then $M^T \Rightarrow M$ in $D([0, 1], \mathcal{S}'(R^d))$. See Section 7.

5. PROOFS

We first introduce some notation. Consider the supercritical branching random field with immigration, and for $t \leq T$ and $A \in \mathcal{B}(R^d)$ let

$\hat{N}_{t,T}(A)$ = number of particles present at time T whose ancestors at time t had positions in A .

Similarly, for $x \in R^d$, $s \leq t \leq T$, and $A \in \mathcal{B}(R^d)$, let

$$\hat{N}_{x,s,t,T}(A) = \text{number of particles present at time } T \text{ whose ancestors at time } t \text{ had positions in } A, \text{ starting with a single particle born at the point } x \text{ at time } s. (\hat{N}_{x,s,t,T} = 0 \text{ for } s > t.) \quad (5.1)$$

Then,

$$\hat{N}_{t,T} = \sum_i \hat{N}_{p_i,0,t,T} + \sum_i \hat{N}_{q_i,u_i,t,T}, \quad (5.2)$$

where $\{p_i\}_{i=1,2,\dots}$ are the points of the initial Poisson field (with intensity γ in B) and $\{(q_i, u_i)\}_{i=1,2,\dots}$ are the points of the immigration Poisson field (with intensity β in \mathcal{C}).

Now we put the new weights to the atoms of $\hat{N}_{t,T}$ by multiplying $\hat{N}_{x,s,t,T}$ by $e^{-\alpha(T-s)}$ and, denoting

$$N_{x,s,t,T} = e^{-\alpha(T-s)} \hat{N}_{x,s,t,T} \quad (5.3)$$

and $N_{t,T}$ the random measure so defined, we have from (5.2)

$$N_{t,T} = \sum_i N_{p_i,0,t,T} + \sum_i N_{q_i,u_i,t,T}. \quad (5.4)$$

The following calculations are valid for both the general model and the special one. It is assumed in these calculations that $N_{t,T}$ takes values in $\mathcal{S}'(R^d)$. For the general model this follows from the boundedness of the initial and immigration sets, which implies that $N_{t,T}$ is finite on bounded sets. For the special model see Corollary 5.4.

The joint characteristic function of the random variables

$$\langle N_{t_1,T}, \phi_1 \rangle, \dots, \langle N_{t_m,T}, \phi_m \rangle, \quad t_1 < \dots < t_m, \quad \phi_1, \dots, \phi_m \in \mathcal{S}(R^d)$$

can be obtained from (5.4) by the usual method [8]:

$$\begin{aligned} & E \exp i \sum_{j=1}^m u_j \langle N_{t_j,T}, \phi_j \rangle \\ &= \exp \left\{ \gamma \int_B \left(E \exp i \sum_{j=1}^m u_j \langle N_{x,0,t_j,T}, \phi_j \rangle - 1 \right) dx \right. \\ &\quad \left. + \beta \int \int_{\mathcal{C}} \left(E \exp i \sum_{j=1}^m u_j \langle N_{x,s,t_j,T}, \phi_j \rangle - 1 \right) dx ds \right\}, \\ &\quad u_1, \dots, u_m \in R. \end{aligned} \quad (5.5)$$

Differentiating (5.5) with respect to the u_i in the usual way we find

$$E\langle N_{t,T}, \phi \rangle = \gamma \int_B E\langle N_{x,0,t,T}, \phi \rangle dx + \beta \int \int_{\mathcal{Q}} E\langle N_{x,s,t,T}, \phi \rangle dx ds, \quad (5.6)$$

and

$$\begin{aligned} \text{Cov}(\langle N_{s,T}, \phi \rangle, \langle N_{t,T}, \psi \rangle) &= \gamma \int_B E\langle N_{x,0,s,T}, \phi \rangle \langle N_{x,0,t,T}, \psi \rangle dx \\ &\quad + \beta \int \int_{\mathcal{Q}} E\langle N_{x,r,s,T}, \phi \rangle \langle N_{x,r,t,T}, \psi \rangle dx dr. \end{aligned} \quad (5.7)$$

Introducing the scaling (2.5) and the change in the immigration intensity ($\beta \rightarrow \beta/T$) and denoting $N_{x,s,t}^T$ and N_t^T the scaled random measures corresponding to $N_{x,s,t,T}$ and $N_{t,T}$, respectively, we have from (5.4)–(5.7),

$$\begin{aligned} E \exp i \sum_{j=1}^m u_j \langle N_{t_j}^T, \phi_j \rangle \\ = \exp \left(T^{bd} \gamma \int_B \left(E \exp i \sum_{j=1}^m u_j \langle N_{x,0,t_j}^T, \phi_j \rangle - 1 \right) dx \right. \\ \left. + T^{bd-1} \beta \int \int_{\mathcal{Q}} \left(E \exp i \sum_{j=1}^m u_j \langle N_{x,s,t_j}^T, \phi_j \rangle - 1 \right) dx d(Ts) \right); \end{aligned}$$

hence,

$$\begin{aligned} E \exp i \sum_{j=1}^m u_j \langle N_{t_j}^T, \phi_j \rangle \\ = \exp \left\{ T^{bd} \left[\gamma \int_B \left(E \exp i \sum_{j=1}^m u_j \langle N_{x,0,t_j}^T, \phi_j \rangle - 1 \right) dx \right. \right. \\ \left. \left. + \beta \int \int_{\mathcal{Q}} \left(E \exp i \sum_{j=1}^m u_j \langle N_{x,s,t_j}^T, \phi_j \rangle - 1 \right) dx ds \right] \right\}, \quad (5.8) \end{aligned}$$

and (skipping the intermediate step involving the time scaling)

$$E\langle N_t^T, \phi \rangle = T^{bd} \left[\gamma \int_B E\langle N_{x,0,t}^T, \phi \rangle dx + \beta \int \int_{\mathcal{Q}} E\langle N_{x,s,t}^T, \phi \rangle dx ds \right] \quad (5.9)$$

and

$$\begin{aligned} \text{Cov}(\langle N_s^T, \phi \rangle, \langle N_t^T, \psi \rangle) \\ = T^{bd} \left[\gamma \int_B E \langle N_{x,0,s}^T, \phi \rangle \langle N_{x,0,t}^T, \psi \rangle dx \right. \\ \left. + \beta \int \int_{\mathcal{C}} E \langle N_{x,r,s}^T, \phi \rangle \langle N_{x,r,t}^T, \psi \rangle dx dr \right]. \end{aligned} \quad (5.10)$$

We consider now the general model. The next lemma contains the basic results we need.

LEMMA 5.1.

$$T^{-bd} E \langle N_t^T, \phi \rangle \rightarrow c \left[\gamma \int_B \mathcal{T}_t \phi(x) dx + \beta \int \int_{\mathcal{C}} \mathcal{T}_{t-s} \phi(x) dx ds \right] \quad (5.11)$$

and

$$\begin{aligned} T^{-bd} \text{Cov}(\langle N_s^T, \phi \rangle, \langle N_t^T, \psi \rangle) \rightarrow \kappa^2 \left[\gamma \int_B \mathcal{T}_s \phi(x) \mathcal{T}_t \psi(x) dx \right. \\ \left. + \beta \int \int_{\mathcal{C}} \mathcal{T}_{s-r} \phi(x) \mathcal{T}_{t-r} \psi(x) dx dr \right] \end{aligned} \quad (5.12)$$

as $T \rightarrow \infty$, the limits being clearly finite.

Proof. This follows from (2.1), (2.2), (5.3) (scaled), (5.9), (5.10), and Corollary 2.2(a). In order to take the limit $T \rightarrow \infty$ inside E in (5.9) and (5.10) we use $|\langle N_{x,s,t}^T, \phi \rangle| \leq \|\phi\|_{\infty} e^{-\alpha T(1-s)} Z_{T(1-s)}^x$, and (2.1), where Z_t^x designates the total number of descendants at time t of a particle initially located at x ; hence we have uniform integrability. To take the limit inside \int_B and $\iint_{\mathcal{C}} \equiv \iint_{\mathcal{C} \wedge t}$, we use this estimate again, (2.3), and the boundedness of B and $\mathcal{C} \wedge t$; hence we have dominated convergence. \square

Proof of Theorem 4.1. Let

$$Q = c \left[\gamma \int_B \mathcal{T}_t \phi(x) dx + \beta \int \int_{\mathcal{C}} \mathcal{T}_{t-r} \phi(x) dx dr \right].$$

Then,

$$\begin{aligned} E|T^{-bd} \langle N_t^T, \phi \rangle - Q|^2 \\ = T^{-2bd} \text{Var} \langle N_t^T, \phi \rangle + (T^{-bd} E \langle N_t^T, \phi \rangle)^2 - 2QT^{-bd} E \langle N_t^T, \phi \rangle + Q^2, \end{aligned}$$

which converges to 0 as $T \rightarrow \infty$ by (5.11) and (5.12). \square

Proof of Theorem 4.2. From (5.8), (5.9), and (5.10), expanding to second order the characteristic functions inside the integrals in (5.8) we have

$$\begin{aligned}
 & E \exp i \sum_{j=1}^m u_j \langle M_{t_j}^T, \phi_j \rangle \\
 &= \exp \left[-iT^{-bd/2} \sum_{j=1}^m u_j E \langle N_{t_j}^T, \phi_j \rangle \right] \\
 & \quad \times \exp \left\{ T^{bd} \left[\gamma \int_B \left(E \exp iT^{-bd/2} \sum_{j=1}^m u_j \langle N_{x,0,t_j}^T, \phi_j \rangle - 1 \right) dx \right. \right. \\
 & \quad \left. \left. + \beta \int \int_{\mathcal{C}} \left(E \exp iT^{-bd/2} \sum_{j=1}^m u_j \langle N_{x,s,t_j}^T, \phi_j \rangle - 1 \right) dx ds \right] \right\} \\
 &= \exp \left[-\frac{1}{2} \sum_{j,k=1}^m u_j u_k T^{-bd} \text{Cov}(\langle N_{t_j}^T, \phi_j \rangle, \langle N_{t_k}^T, \phi_k \rangle) \right] \\
 & \quad \times \exp \left\{ T^{bd} \left[\gamma \int_B J_1^T(x) dx + \beta \int \int_{\mathcal{C} \wedge t_m} J_2^T(x, s) dx ds \right] \right\}, \quad (5.13)
 \end{aligned}$$

where $J_1^T(x)$ and $J_2^T(x, s)$ are the (second order) error terms in the expansions. The limit as $T \rightarrow \infty$ of the first exponential in (5.13) gives the desired result by (5.12); hence we must show that the exponent in the second exponential tends to 0 as $T \rightarrow \infty$. We have

$$|J_1^T(x)| = (T^{-bd}/2) K_1^T(x) \quad \text{and} \quad |J_2^T(x, s)| = (T^{-bd}/2) K_2^T(x, s),$$

where $K_1^T(x) \rightarrow 0$, $K_2^T(x, s) \rightarrow 0$ as $T \rightarrow \infty$, and

$$\begin{aligned}
 K_1^T(x) &\leq 2E \left(\sum_{j=1}^m u_j \langle N_{x,0,t_j}^T, \phi_j \rangle \right)^2 \leq L e^{-2\alpha T} E Z_T^2, \\
 K_2^T(x, s) &\leq 2E \left(\sum_{j=1}^m u_j \langle N_{x,s,t_j}^T, \phi_j \rangle \right)^2 \leq L e^{-2\alpha T(1-s)} E Z_{T(1-s)}^2,
 \end{aligned}$$

where $L < \infty$ is a constant; hence, by (2.3) and the boundedness of B and $\mathcal{C} \wedge t$ we can take the limit inside the integrals by dominated convergence.

The existence of the $\mathcal{S}'(R^d)$ -valued Gaussian process M can be proved by Kolmogorov's consistency theorem, and it follows also from the representation in Theorem 4.3 (1). \square

Proof of Theorem 4.3. (1) The stochastic integrals are well defined because

$$\int_B (\mathcal{T}_t \phi(x))^2 dx < \infty \quad \text{and} \quad \int \int_{\mathcal{C}} (\mathcal{T}_{t-} \phi(x))^2 dx dr < \infty,$$

which holds due to the assumptions on $\{\mathcal{T}_t\}$ and the boundedness of B and $\mathcal{C} \wedge t$, and they are linear and L^2 -continuous in ϕ ; hence they are $\mathcal{S}'(R^d)$ -valued Gaussian random elements (see [10, 33]); therefore to verify the representation it suffices to observe that the mean and covariance functions are right.

(2) This follows from the definitions and known results (see [28, 33]).

(3) The covariance functional of M satisfies (2.9) with $\mathcal{R}_t = \mathcal{T}_t$; hence by Theorem 2.3 M is Markovian and obeys the Langevin equation $dM = \mathcal{G}^* M dt + d\mathcal{W}_t$, where \mathcal{W} is an $\mathcal{S}'(R^d)$ -Wiener process with covariance functional

$$\begin{aligned}
 & \text{Cov}(\langle \mathcal{W}_s, \phi \rangle, \langle \mathcal{W}_t, \psi \rangle) \\
 &= \kappa^2 \left\{ \gamma \int_B \mathcal{T}_s \phi(x) \mathcal{T}_s \psi(x) dx + \beta \int \int_{\mathcal{C}} \mathcal{T}_{s-r} \phi(x) \mathcal{T}_{s-r} \psi(x) dx dr \right. \\
 & \quad - \gamma \int_B \phi(x) \psi(x) dx \\
 & \quad - \int_0^s \left[\gamma \int_B \mathcal{T}_u \mathcal{G} \phi(x) \mathcal{T}_u \psi(x) dx + \beta \int \int_{\mathcal{C}} \mathcal{T}_{u-r} \mathcal{G} \phi(x) \mathcal{T}_{u-r} \psi(x) dx dr \right. \\
 & \quad \left. \left. + \gamma \int_B \mathcal{T}_u \phi(x) \mathcal{T}_u \mathcal{G} \psi(x) dx + \beta \int \int_{\mathcal{C}} \mathcal{T}_{u-r} \phi(x) \mathcal{T}_{u-r} \mathcal{G} \psi(x) dx dr \right] du \right\} \\
 &= \kappa^2 \left\{ \gamma \int_B \mathcal{T}_s \phi(x) \mathcal{T}_s \psi(x) dx + \beta \int \int_{\mathcal{C}} \mathcal{T}_{s-r} \phi(x) \mathcal{T}_{s-r} \psi(x) dx dr \right. \\
 & \quad - \gamma \int_B \phi(x) \psi(x) dx - \gamma \int_B \int_0^s \frac{d}{du} (\mathcal{T}_u \phi(x) \mathcal{T}_u \psi(x)) du dx \\
 & \quad \left. - \beta \int \int_{\mathcal{C}} \int_0^s \frac{d}{du} (\mathcal{T}_{u-r} \phi(x) \mathcal{T}_{u-r} \psi(x)) du dx dr \right\} \\
 &= \kappa^2 \beta \int \int_{\mathcal{C}} \phi(x) \psi(x) \chi_{[0, s]}(r) dx dr \\
 &= \kappa^2 \beta \int_0^s \int_{\mathcal{C}} \phi(x) \psi(x) dx dr,
 \end{aligned}$$

for $s \leq t$. We have used (2.10). Hence $\mathcal{W} = \kappa \beta^{1/2} \mathcal{W}^{\mathcal{C}}$.

That $\mathcal{W}^{\mathcal{C}}$ is an $\mathcal{S}'(R^d)$ -valued square-integrable martingale from some p follows from Theorem 2.3 and the proof of (5). Since $E \langle \mathcal{W}_t^{\mathcal{C}}, \phi \rangle^2 \leq L t \int_{R^d} \phi^2(x) dx$, where $L < \infty$ is a constant, we may take $p > d/4$.

The independence of M_0 (or W^B) and $\mathcal{W}^{\mathcal{C}}$ follows from the fact that these random elements depend only on the systems generated by the initial field and the immigration field, respectively, and these fields are independent.

(4) It suffices to verify that the evolution solution has the right mean and covariance functionals; this can be done using the formal expression $E\langle d\mathcal{W}_s^{\mathcal{C}}, \phi \rangle \langle d\mathcal{W}_t^{\mathcal{C}}, \psi \rangle = \delta(t-s) \int_{\mathcal{C}} \psi(x) \psi(x) dx ds dt$.

(5) By (3) and Theorem 2.3, $\langle M_t, \phi \rangle - \int_0^t \langle M_s, \mathcal{G}\phi \rangle ds$, $t \geq 0$, is a square-integrable martingale for each $\phi \in \mathcal{S}(R^d)$. Hence for each $0 < \tau < \infty$ we have, by Doob's inequality and other standard inequalities,

$$\begin{aligned} & \left[E \sup_{0 \leq t \leq \tau} \left(\langle M_t, \phi \rangle - \int_0^t \langle M_s, \mathcal{G}\phi \rangle ds \right)^2 \right]^{1/2} \\ & \leq 2 \left[E \left(\langle M_\tau, \phi \rangle - \int_0^\tau \langle M_s, \mathcal{G}\phi \rangle ds \right)^2 \right]^{1/2} \\ & \leq 2 \left\{ \left[E \langle M_\tau, \phi \rangle^2 \right]^{1/2} + \left[E \left(\int_0^\tau \langle M_s, \mathcal{G}\phi \rangle ds \right)^2 \right]^{1/2} \right\} \\ & \leq 2 \left\{ \left[E \langle M_\tau, \phi \rangle^2 \right]^{1/2} + \tau^{1/2} \left[\int_0^\tau E \langle M_s, \mathcal{G}\phi \rangle^2 ds \right]^{1/2} \right\}; \end{aligned}$$

therefore

$$\begin{aligned} \left[E \sup_{0 \leq t \leq \tau} \langle M_t, \phi \rangle^2 \right]^{1/2} & \leq 2 \left\{ \left[E \sup_{0 \leq t \leq \tau} \left(\langle M_t, \phi \rangle - \int_0^t \langle M_s, \mathcal{G}\phi \rangle ds \right)^2 \right]^{1/2} \right. \\ & \quad \left. + \left[E \sup_{0 \leq t \leq \tau} \left(\int_0^t \langle M_s, \mathcal{G}\phi \rangle ds \right)^2 \right]^{1/2} \right\} \\ & \leq 4 \left[E \langle M_\tau, \phi \rangle^2 \right]^{1/2} + 6\tau^{1/2} \left[\int_0^\tau E \langle M_s, \mathcal{G}\phi \rangle^2 ds \right]^{1/2}. \end{aligned}$$

Now, since \mathcal{T}_t is a contraction with respect to $\|\cdot\|_\infty$, from (4.1) and the fact that B and $\mathcal{C} \wedge t$ have finite Lebesgue measure we have

$$\max(E \langle M_t, \phi \rangle^2, E \langle M_t, \mathcal{G}\phi \rangle^2) \leq K(\lambda_d(B) + \lambda_{d+1}(\mathcal{C} \wedge t))$$

for all t , where $K < \infty$ is a constant depending on ϕ , and hence there is a positive locally bounded function f on $[0, \infty)$ such that $\sup_{\tau \geq 0} (E \sup_{0 \leq t \leq \tau} \langle M_t, \phi \rangle^2) / f(\tau) < \infty$ for all $\phi \in \mathcal{S}(R^d)$. The conclusion follows by [30].

(6) Using the representation (1) we obtain

$$\langle M_t, \phi \rangle = \int_{R^d} \tilde{\phi}(\lambda) Z_t(d\lambda),$$

where $\tilde{\phi}$ is the Fourier transform of ϕ and

$$\begin{aligned} Z_t(d\lambda) & = \kappa \left[\gamma^{1/2} \int_{R^d} \int_{R^d} (2\pi)^{-d/2} e^{i\lambda \cdot y} p_t(x, dy) W_1(dx) \right. \\ & \quad \left. + \beta^{1/2} \int_{R^d} \int_{R^d} \int_{\mathcal{C}} (2\pi)^{-d/2} e^{i\lambda \cdot y} p_{t-r}(x, dy) W_2(dx, dr) \right] d\lambda, \end{aligned}$$

where \cdot stands for the inner product in R^d . Then $\sigma_t(d\lambda)$ is computed from $\sigma_t(A) = E|Z_t(A)|^2$, $A \in \mathcal{B}(R^d)$ (see [10, 27]).

(7) This follows from Lévy's continuity theorem on nuclear spaces [5]. \square

Proof of Theorem 4.4. (3) This of course is the same as statement (5) of the previous theorem. The proof, however, is not quite the same because now the sets B and $\mathcal{C} \wedge t$ do not have finite Lebesgue measure. The first part of the proof is as before, but now the existence of a positive locally finite function f on $[0, \infty)$ such that

$$\sup_{\tau \geq 0} \left(E \sup_{0 \leq t \leq \tau} \langle M_t, \phi \rangle^2 \right) / f(\tau) < \infty$$

depends on the particular form of $E\langle M_t, \phi \rangle^2$ in the present case; it is easy to see that $E\langle M_t, \phi \rangle^2 \leq K\|\phi\|_\infty \int_{R^d} |\phi(x)| dx (1+t)$.

Proving the rest of the assertions requires only standard calculations involving the Gaussian kernel. \square

We turn to the special model. We recall that in this case \tilde{L} is standard Brownian motion, and therefore $\{\mathcal{T}_t\}$ stands now for the usual Brownian semigroup. We have $b = \frac{1}{2}$ in the scaling (2.5), and the branching law has finite third moment. We shall need several preliminary results for the proofs of Theorems 4.5 and 4.6.

LEMMA 5.2. For $N_{x,r,t,T}$ defined by (5.3), and $\phi, \psi \in \mathcal{S}(R^d)$ we have

$$\int_{R^d} E\langle N_{x,r,t,t}, \phi \rangle dx = \int_{R^d} \phi(x) dx, \quad (5.14)$$

and, for $s \leq t$,

$$\begin{aligned} & \int_{R^d} E\langle N_{x,r,s,s}, \phi \rangle \langle N_{x,r,t,t}, \psi \rangle dx \\ &= e^{-\alpha(s-r)} \left[\int_{R^d} \phi(x) \mathcal{T}_{t-s} \psi(x) dx \right. \\ & \quad \left. + m_2 V \int_0^{s-r} e^{\alpha u} \int_{R^d} \phi(x) \mathcal{T}_{t-s+2u} \psi(x) dx du \right]. \end{aligned} \quad (5.15)$$

Proof. This follows from the basic lemma in [13]. \square

LEMMA 5.3. For $s \leq t \leq T$, and $\phi, \psi \in \mathcal{S}(R^d)$,

$$E\langle N_{t,T}, \phi \rangle = [\gamma + \beta \lambda_1(C \cap [0, t])] \int_{R^d} \phi(x) dx \quad (5.16)$$

and

$$\begin{aligned}
 & \text{Cov}(\langle N_{s,T}, \phi \rangle, \langle N_{t,T}, \psi \rangle) \\
 &= \gamma \left\{ [m_2 e^{-\alpha s} - (m_2 - m_1 + 1) e^{-\alpha T}] (m_1 - 1)^{-1} \int_{R^d} \phi(x) \right. \\
 & \quad \times \mathcal{T}_{t-s} \psi(x) dx + m_2 V e^{-\alpha s} \int_0^s e^{\alpha u} \int_{R^d} \phi(x) \mathcal{T}_{t-s+2u} \psi(x) dx du \Big\} \\
 & + \beta \int_{C \cap [0, s]} \left\{ [m_2 e^{-\alpha(s-r)} - (m_2 - m_1 + 1) e^{-\alpha(T-r)}] (m_1 - 1)^{-1} \right. \\
 & \quad \times \int_{R^d} \phi(x) \mathcal{T}_{t-s} \psi(x) dx + m_2 V e^{-\alpha(s-r)} \int_0^{s-r} e^{\alpha u} \int_{R^d} \phi(x) \\
 & \quad \times \mathcal{T}_{t-s+2u} \psi(x) dx du \Big\} dr. \tag{5.17}
 \end{aligned}$$

COROLLARY 5.4. $N_{t,T}$ takes values in $\mathcal{S}'(R^d)$ a.s.

COROLLARY 5.5. For each $T > 0$, $t \in [0, 1]$, $s \leq t$, and $\phi, \psi \in \mathcal{S}(R^d)$,

$$E\langle N_t^T, \phi \rangle = T^{d/2} [\gamma + \beta \lambda_1(C \cap [0, t])] \int_{R^d} \phi(x) dx, \tag{5.18}$$

$$\begin{aligned}
 & \text{Cov}(\langle N_s^T, \phi \rangle, \langle N_t^T, \psi \rangle) \\
 &= T^{d/2} \left\{ \gamma [m_2 e^{-\alpha Ts} - (m_2 - m_1 + 1) e^{-\alpha T}] (m_1 - 1) \right. \\
 & \quad \times \int_{R^d} \phi(x) \mathcal{T}_{t-s} \psi(x) dx + m_2 V \alpha^{-1} (1 - e^{-\alpha Ts}) \int_0^s F_T(u) \\
 & \quad \times \int_{R^d} \phi(x) \mathcal{T}_{t+s-2u} \psi(x) dx du \Big] \\
 & + \beta \int_{\mathcal{C} \cap [0, s]} \left[(m_2 e^{-\alpha T(s-r)} - (m_2 - m_1 + 1) e^{-\alpha T(1-r)}) \right. \\
 & \quad \times (m_1 - 1)^{-1} \int_{R^d} \phi(x) \mathcal{T}_{t-s} \psi(x) dx \\
 & \quad + m_2 V \alpha^{-1} (1 - e^{-\alpha T(s-r)}) \int_0^{s-r} G_T(u, r) \\
 & \quad \times \int_{R^d} \phi(x) \mathcal{T}_{t+s-2r-2u} \psi(x) dx du \Big] dr \Big\}, \tag{5.19}
 \end{aligned}$$

where

$$\begin{aligned}
 F_T(u) &= \alpha T e^{-\alpha T u} (1 - e^{-\alpha Ts})^{-1} \quad \text{and} \\
 G_T(u, r) &= \alpha T e^{-\alpha T u} (1 - e^{-\alpha T(u-r)})^{-1}, \tag{5.20}
 \end{aligned}$$

and there is a constant $K < \infty$ depending on ϕ such that

$$E\langle M_t^T, \phi \rangle^2 < K. \quad (5.21)$$

COROLLARY 5.6. For $t \in [0, 1]$, $s \leq t$, and $\phi, \psi \in \mathcal{S}(R^d)$,

$$T^{-d/2} E\langle N_t^T, \phi \rangle \rightarrow [\gamma + \beta \lambda_1(C \cap [0, t])] \int_{R^d} \phi(x) dx \quad (5.22)$$

(trivial by (5.18)), and

$$\begin{aligned} T^{-d/2} \text{Cov}(\langle N_s^T, \phi \rangle, \langle N_t^T, \psi \rangle) \\ \rightarrow m_2(m_1 - 1)^{-1} \left[\gamma \int_{R^d} \mathcal{T}_s \phi(x) \mathcal{T}_t \psi(x) dx \right. \\ \left. + \beta \int_{C \cap [0, s]} \int_{R^d} \mathcal{T}_{s-r} \phi(x) \mathcal{T}_{t-r} \psi(x) dx dr \right] \end{aligned} \quad (5.23)$$

as $T \rightarrow \infty$.

LEMMA 5.7. There is a constant $K < \infty$ such that

$$\int_{R^d} E|\langle N_{x,s,t}^T, \phi \rangle|^3 dx \leq K T^{d/2} \int_{R^d} |\phi(x)|^3 dx. \quad (5.24)$$

Remark. Corollary 5.6 is the same as Lemma 5.1 for the special case, but the proofs are entirely different. For Lemma 5.1 we relied on the general Theorem 2.1, and for Corollary 5.6 we used the basic lemma in [13].

Proofs of Lemma 5.3, Corollaries 5.4 to 5.6 and Lemma 5.7. From (5.1) we have

$$\langle \hat{N}_{x,r,t,T}, \phi \rangle = \sum_i \phi(x_i) Z_{i,T}^{x_i}, \quad (a)$$

where $\{x_i\}_i$ are the locations of the atoms of $\hat{N}_{x,s,t,T}$ and $Z_{i,T}^y$ is the total number of descendants at time T of a particle located at the point y at time t ; in particular

$$\langle \hat{N}_{x,r,t,t}, \phi \rangle = \sum_i \phi(x_i). \quad (b)$$

Let E_t denote the conditional expectation given the σ -algebra generated by all the genealogies (including the initial positions of the initial and the immigrant particles) and the migrations of the particles up to time t . Then, using the Markov and independence properties of the system, and (2.4), (a), and (b), we have

$$\begin{aligned} E\langle \hat{N}_{x,r,t,T}, \phi \rangle &= E \sum_i \phi(x_i) E_t Z_{i,T}^{x_i} = e^{\alpha(T-t)} E \sum_i \phi(x_i) \\ &= e^{\alpha(T-t)} E\langle \hat{N}_{x,r,t,t}, \phi \rangle; \end{aligned}$$

hence, by (5.3),

$$E\langle N_{x,r,t,T}, \phi \rangle = E\langle N_{x,r,t,t}, \phi \rangle,$$

and therefore, by (5.14) we have

$$\int_{R^d} E \langle N_{x,r,t,T}, \phi \rangle dx = \int_{R^d} \phi(x) dx. \quad (c)$$

Let $s \leq t$ and denote by $\{x_i\}_i$ the locations of the atoms of $\hat{N}_{x,r,s,T}$, $\{y_j\}_j$ those of the atoms of $\hat{N}_{x,r,t,T}$, and x_{ij} the location of the j th descendant at time t of the particle located at x_i at time s ; therefore

$$Z_{s,T}^{x_i} = \sum_j Z_{t,T}^{x_{ij}}. \quad (d)$$

Using E_t as above, (2.4), (a), (b), and (d),

$$\begin{aligned} & E \langle \hat{N}_{x,r,s,T}, \phi \rangle \langle \hat{N}_{x,r,t,T}, \psi \rangle \\ &= E \sum_i \phi(x_i) Z_{s,T}^{x_i} \sum_j \psi(y_j) Z_{t,T}^{y_j} \\ &= E \sum_i \phi(x_i) \sum_k Z_{t,T}^{x_{ik}} \sum_j \psi(y_j) Z_{t,T}^{y_j} \\ &= E \sum_i \phi(x_i) \sum_k Z_{t,T}^{x_{ik}} \left(\sum_j \psi(x_{ij}) Z_{t,T}^{x_{ij}} + \sum_{l \neq j} \sum_j \psi(x_{lj}) Z_{t,T}^{x_{lj}} \right) \\ &= E \sum_i \phi(x_i) \sum_j \psi(x_{ij}) \sum_k E_t Z_{t,T}^{x_{ik}} Z_{t,T}^{x_{ij}} \\ &\quad + E \sum_i \phi(x_i) \sum_{l \neq i} \sum_j \psi(x_{lj}) \sum_k E_t Z_{t,T}^{x_{ik}} Z_{t,T}^{x_{lj}} \\ &= E \sum_i \phi(x_i) \sum_j \psi(x_{ij}) \left[E_t (Z_{t,T}^{x_{ij}})^2 + \sum_{k \neq j} E_t Z_{t,T}^{x_{ik}} E_t Z_{t,T}^{x_{lj}} \right] \\ &\quad + E \sum_i \phi(x_i) \sum_{l \neq i} \sum_j \psi(x_{lj}) \sum_k E_t Z_{t,T}^{x_{ik}} E_t Z_{t,T}^{x_{lj}} \\ &= E \sum_i \phi(x_i) \sum_j \psi(x_{ij}) \left\{ [m_2 e^{2\alpha(T-t)} - (m_2 - m_1 + 1) e^{\alpha(T-t)}] \right. \\ &\quad \left. \times (m_1 - 1)^{-1} + e^{2\alpha(T-t)} (Z_{s,T}^{x_i} - 1) \right\} \\ &\quad + e^{2\alpha(T-t)} E \sum_i \phi(x_i) Z_{s,t}^{x_i} \sum_{l \neq i} \sum_j \psi(x_{lj}) \\ &= (m_2 - m_1 + 1) (m_1 - 1)^{-1} (e^{2\alpha(T-t)} - e^{\alpha(T-t)}) E \sum_i \phi(x_i) \sum_j \psi(x_{ij}) \\ &\quad + e^{2\alpha(T-t)} E \sum_i \phi(x_i) Z_{s,t}^{x_i} \sum_j \psi(y_j). \quad (e) \end{aligned}$$

To compute the expectations in the last part of (e) let $E_{s,t}$ denote the conditional expectation given the σ -algebra generated by all the genealogies up to time t (including initial positions) and the migrations up to time s ;

then, since the migrations along the branches are Brownian motions independently of the branchings, using (a) we have

$$\begin{aligned}
 E \sum_i \phi(x_i) \sum_j \psi(x_{ij}) &= E \sum_i \phi(x_i) \sum_j E_{s,t} \psi(x_{ij}) \\
 &= E \sum_i \phi(x_i) Z_{s,t}^{x_i} \mathcal{T}_{t-s} \psi(x_i) \\
 &= E \langle \hat{N}_{x,r,s,t}, \phi \mathcal{T}_{t-s} \psi \rangle
 \end{aligned} \tag{f}$$

and

$$\begin{aligned}
 E \sum_i \phi(x_i) Z_{s,t}^{x_i} \sum_j \psi(y_j) &= E \sum_i \phi(x_i) Z_{s,t}^{x_i} \sum_i \sum_j E_{s,t} \psi(x_{ij}) \\
 &= E \sum_i \phi(x_i) Z_{s,t}^{x_i} \sum_i Z_{s,t}^{x_i} \mathcal{T}_{t-s} \psi(x_i) \\
 &= E \langle \hat{N}_{x,r,s,t}, \phi \rangle \langle \hat{N}_{x,r,s,t}, \mathcal{T}_{t-s} \psi \rangle.
 \end{aligned} \tag{g}$$

Substituting (f) and (g) into (e) and integrating, and then using (5.3) and (c),

$$\begin{aligned}
 &\int_{R^d} E \langle \hat{N}_{x,r,s,t}, \phi \rangle \langle \hat{N}_{x,r,t,T}, \psi \rangle dx \\
 &= (m_2 - m_1 + 1)(m_1 - 1)^{-1} (e^{2\alpha(T-t)} - e^{\alpha(T-t)}) \\
 &\quad \times \int_{R^d} E \langle \hat{N}_{x,r,s,t}, \phi \mathcal{T}_{t-s} \psi \rangle dx \\
 &\quad + e^{2\alpha(T-t)} \int_{R^d} E \langle \hat{N}_{x,r,s,t}, \phi \rangle \langle \hat{N}_{x,r,s,t}, \mathcal{T}_{t-s} \psi \rangle dx \\
 &= (m_2 - m_1 + 1)(m_1 - 1)^{-1} (e^{2\alpha(T-t)} - e^{\alpha(T-t)}) e^{\alpha(t-r)} \\
 &\quad \times \int_{R^d} \phi(x) \mathcal{T}_{t-s} \psi(x) dx \\
 &\quad + e^{2\alpha(T-t)} \int_{R^d} E \langle \hat{N}_{x,r,s,t}, \phi \rangle \langle \hat{N}_{x,r,s,t}, \mathcal{T}_{t-s} \psi \rangle dx.
 \end{aligned} \tag{h}$$

Letting $t = s$ and $T = t$ in (h) and using (5.3) and (5.15) we get

$$\begin{aligned}
 &\int_{R^d} E \langle \hat{N}_{x,r,s,t}, \phi \rangle \langle \hat{N}_{x,r,s,t}, \psi \rangle dx \\
 &= (m_2 - m_1 + 1)(m_1 - 1)^{-1} (e^{2\alpha(t-s)} - e^{\alpha(t-s)}) e^{\alpha(s-r)} \int_{R^d} \phi(x) \psi(x) dx \\
 &\quad + e^{\alpha(2t-r-s)} \left[\int_{R^d} \phi(x) \psi(x) dx + m_2 V \int_0^{s-r} e^{\alpha u} \int_{R^d} \phi(x) \mathcal{T}_{2u} \psi(x) dx du \right].
 \end{aligned} \tag{i}$$

Substituting (i) into (h) and using (5.3),

$$\begin{aligned}
 & \int_{R^d} E \langle N_{x,r,s,T}, \phi \rangle \langle N_{x,r,t,T}, \psi \rangle dx \\
 &= [m_2 e^{-\alpha(s-r)} - (m_2 - m_1 + 1) e^{-\alpha(T-r)}] \\
 & \quad \times (m_1 - 1)^{-1} \int_{R^d} \phi(x) \mathcal{T}_{t-s} \psi(x) dx \\
 & \quad + m_2 V e^{-\alpha(s-r)} \int_0^{s-r} e^{\alpha u} \int_{R^d} \phi(x) \mathcal{T}_{t-s+2u} \psi(x) dx du. \quad (j)
 \end{aligned}$$

Finally, substituting (c) and (j) into (5.6) and (5.7) with $B = R^d$ and $\mathcal{C} = R^d \times C$ we obtain (5.16) and (5.17), so Lemma 5.3 is proved.

From (5.16) we have $E \langle N_{t,T}, \phi \rangle < \infty$ for $\phi(x) = (1 + \|x\|^2)^{-p}$ with $p > d/2$; hence $N_{t,T}$ is a.s. a random tempered Radon measure, proving Corollary 5.4.

Introducing the scaling $(x, t, \beta) \rightarrow (T^{1/2}x, Tt, \beta/T)$ into (5.16) and (5.17) we get (5.18) and (5.19). Note that (5.19) can be written more simply; we have introduced the functions F_T and G_T in preparation for the proof of (5.23). Inequality (5.21) follows from (5.19) by standard estimates. Hence Corollary 5.5 is proved.

We have already noted that (5.22) is trivial. Observing that both $F_T(u)$ and $G_T(u, r)$ given in (5.20) tend to the delta function $\delta(u)$ as $T \rightarrow \infty$ in the intervals where they are defined, we obtain (5.23). Corollary 5.6 is proved.

The proof of Lemma 5.7 again starts from (a), using E_t as above, (2.3) (5.3), (b), (c), the scaling and standard estimates. Note that the finite third moment assumption of the branching law is used here. \square

Proof of Theorems 4.5 and 4.6. The law of large numbers is proved the same way as Theorem 4.1, using (5.22) and (5.23). The proof of $M^T \Rightarrow_f M$ is essentially the same as that in Theorem 4.2, using (5.8), (5.9), (5.10), (5.22), and (5.23), except that we must take the third order error terms in the expansions of the characteristic functions inside the integrals in (5.8); the argument of Theorem 4.2 cannot be used here because the sets B and $\mathcal{C} \wedge t$ have infinite Lebesgue measure. Hence we have (5.13) with error

$$\gamma \int_{R^d} J_1^T(x) dx + \beta \int_{R^d} \int_{C \cap [0, t]} J_2^T(x, s) dx ds,$$

where

$$|J_1^T(x)| \leq (T^{-d/4})^3 / 3! E \left| \sum_{j=1}^m u_j \langle N_{x,0,t_j}^T, \phi_j \rangle \right|^3,$$

and a similar bound for $J_2^T(x, s)$. Then by (5.24), $\int_{R^d} J_1^T(x) dx \rightarrow 0$ as $T \rightarrow \infty$ and similarly for the other term.

That M^T can be realized in $D([0, 1], \mathcal{S}'(R^d))$ follows from the fact that the special model is Markovian and general results about Markov processes. \square

6. EXAMPLE

This example is relevant in particle transport theory and polymer chemistry. Suppose the particles in the general model migrate in R^d , $d \geq 3$, in the following way. The initial and the immigrant particles follow arbitrary directions with constant speed 1 during G -distributed random times, the offspring particles independently choose directions with radial symmetry forming i.i.d. angles θ (not supported in $\{0, \pi\}$) with respect to their parent particle's direction, and they also move with constant speed 1 starting from the branching point, etc. Suppose G has a finite moment of order $3 + \delta$ for some $\delta > 0$. Then the convergence condition (2.7) holds with $b = \frac{1}{2}$ in the scaling (2.5), the limit L being Brownian motion on R^d with variance parameter

$$\sigma^2 = (E\tau)^{-1} [\text{Var } \tau + (E\tau)^2(1 + E \cos \theta)(1 - E \cos \theta)^{-1}],$$

where τ has distribution G . This is proved in [11] (see also [26]).

From Theorems 4.1, 4.2, and 4.4 we have

$$\begin{aligned} T^{-d/2} \langle N_t^T, \phi \rangle &\rightarrow c \left[\gamma \int_B \left(\int_{R^d} \phi(y) e^{-\|y-x\|^2/2\tilde{\sigma}^2 t} (2\pi\tilde{\sigma}^2 t)^{-d/2} dy \right) dx \right. \\ &\quad \left. + \beta \int \int_{\mathcal{G} \wedge t} \left(\int_{R^d} \phi(y) e^{-\|y-x\|^2/2\tilde{\sigma}^2(t-r)} (2\pi\tilde{\sigma}^2(t-r))^{-d/2} dy \right) dx dr \right] \end{aligned}$$

in L^2 as $T \rightarrow \infty$, and $T^{-d/4}(N^T - EN^T) \Rightarrow_f M$ as $T \rightarrow \infty$, where M is a generalized Ornstein-Uhlenbeck process satisfying

$$\begin{aligned} dM &= \frac{\tilde{\sigma}^2}{2} \Delta M dt + \kappa \beta^{1/2} d\mathcal{W}^{\mathcal{G}}, \\ M_0 &= \kappa \gamma^{1/2} W^B, \end{aligned}$$

and the other properties given in Theorem 4.4, where

$$\tilde{\sigma}^2 = (E\tilde{\tau})^{-1} [\text{Var } \tilde{\tau} + (E\tilde{\tau})^2(1 + E \cos \theta)(1 - E \cos \theta)^{-1}],$$

$\tilde{\tau}$ having distribution $\tilde{G}(dt) = m_1 e^{-\alpha t} G(dt)$.

In polymer chemistry τ and θ are deterministic. Our hypotheses on G exclude deterministic τ , but analogous results can be proved also in this case.

7. REMARKS

The papers that first motivated our interest in the present problem are [22] (which led to [18, 19]), and [9] for the infinite particle case.

Our results are valid also in the Galton–Watson case, replacing κ in (2.2) by $E\xi^2$, where ξ is the L^2 -limit as $n \rightarrow \infty$ of $m_1^{-n}\xi_n$, ξ_n being the number of particles in the n th generation of the underlying branching process. In this case $L = \tilde{L}$ in Theorem 2.1.

It would be desirable to remove the boundedness restrictions on the initial and the immigration sets in the general model, and to prove results of the present type for the supercritical case with $p_0 > 0$, and for the critical and subcritical cases, but it is not clear how to deal with disappearing particles. The idea of “backward trees” [24] for the critical case seems relevant. Other interesting extensions would be models with particles of several types, random environments, interactions, boundaries, controls.

A more general approach is to describe the present model by a random point measure on a function space which counts the trajectories of the system in Borel sets of functions. The appropriate function space is $D([0, T], R^d \cup \{\delta\})$, where δ is a place where particles live before they are born and after they die. This was done in [7, 12] for the supercritical case with $p_0 = 0$ and no immigration. This needs defining nuclear spaces of test functions and distributions on spaces like $D([0, T], R^d \cup \{\delta\})$, and seeking a Lévy-type continuity theorem for such spaces; see [7] for a brief discussion of this point, and [32], where a similar problem arises.

For the special model it should be possible to prove relative weak compactness of $\{M^T\}_{T \geq 1}$ in $D([0, 1], \mathcal{S}'(R^d))$ using martingale methods [20, 23, 29]. The relevant martingale for the process $\{N_{t,T}, 0 \leq t \leq T\}$ is given in the next result (there is a corresponding martingale for the centered process).

THEOREM 7.1. *Let \mathcal{F}_t denote the σ -algebra generated by the system up to time $t \leq T$ (i.e., by all $N_{p_i, 0, s, T}$ and all $N_{q_i, u_i, s, T}$ with $s \leq t$; see (5.4)), and $Z_{t,T}^x$ the total number of descendants at time T of a particle located at x at time t . Then for each $\phi \in \mathcal{S}(R^d)$,*

$$\begin{aligned} \langle N_{t,T}, \phi \rangle - \int_0^t \langle N_{r,T}, \tfrac{1}{2} \Delta \phi \rangle dr - \sum_{i, u_i \leq t} \phi(q_i) e^{-\alpha(T-u_i)} Z_{u_i, T}^{q_i}, \\ 0 \leq t \leq T, \end{aligned} \quad (7.1)$$

is a martingale with respect to $\{\mathcal{F}_t\}$ with increasing process

$$\int_0^t \sum_i \|\nabla \phi(x_i)\|^2 e^{-2\alpha(T-s_i)} (Z_{r,T}^{x_i})^2 dr, \quad 0 \leq t \leq T, \quad (7.2)$$

where $\{x_i\}_i$ are the locations of the atoms of $N_{r,T}$ and $\{s_i\}_i$ are the times of birth of their corresponding first ancestors.

The proof of this theorem is quite long; a sketch is contained in [16]. Observe that in the case without immigration the martingale (7.1) is the same as in the case where the particles perform Brownian motions but do not branch; this led in [7] to the wrong conclusion that the increasing process of the martingale (7.1) should also be the same as if the particles did not branch, i.e., $\int_0^t \langle N_{r,T}, \|\nabla \phi\|^2 \rangle dr$ instead of (7.2) with $s_i = 0$ for all i .

By [31], to prove tightness of $\{M^T\}_{T \geq 1}$ in $D([0, 1], \mathcal{S}'(R^d))$ it suffices to show tightness of $\{\langle M^T, \phi \rangle\}_{T \geq 1}$ in $D([0, 1], R)$ for each $\phi \in \mathcal{S}(R^d)$. An alternative way of proving $M^{\tilde{T}} \Rightarrow M$ is the following: prove tightness of $\{M^T\}_{T \geq 1}$ in $D([0, 1], \mathcal{S}'(R^d))$ and show that $\tilde{M}^T \Rightarrow \tilde{M}$ in $\mathcal{S}'(R^{d+1})$, where \tilde{x} is defined for $x \in D([0, 1], \mathcal{S}'(R^d))$ by $\langle \tilde{x}, \Phi \rangle = \int_0^1 \langle x_t, \Phi(\cdot, t) \rangle dt$, $\Phi \in \mathcal{S}(R^{d+1})$ [4].

Finally, let us observe the difference between the present fluctuation limit results and those for the usual process [3, 13–15]. This can be done by comparing the Langevin equations. In the usual case, according to each rescaling the evolution term contains the effect of the particle motion ($\frac{1}{2}\Delta$) and/or the Malthusian growth (α), and the noise process \mathcal{W} is a time-inhomogeneous generalized Wiener process containing the effects of the motion, the lifetime and the branching; in particular $d\mathcal{W}$ is in general more complex than a space-time white noise. In the present case the evolution term also represents the particle motion but does not contain a Malthusian growth part, and $d\mathcal{W}$ is a space-time white noise on the immigration set \mathcal{C} ; the noise process \mathcal{W} is caused only by the immigration of particles, and the possible time-inhomogeneity of it is due only to the shape of \mathcal{C} and not to other causes as in the usual case; the branching has a deterministic effect in the limit due to the almost-sure invariance principle [18], and it appears only through the constant κ in (2.2).

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Note added in proof. The method of proof for the special model can be modified in order to allow arbitrary initial and immigration sets.

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